New constructions of Kakeya and Besicovitch sets

Yuval Peres [1]

Based on work with

Y. Babichenko, R. Peretz, P. Sousi, P. Winkler and

the forthcoming book *Fractals in Probability and Analysis* with C. Bishop

[1] Microsoft Research

Besicovitch sets – History

A subset $S \subseteq \mathbb{R}^2$ is called a **Besicovitch** set if it contains a unit segment in every direction. It is called a **Kakeya** set if a unit segment can be rotated 360 degrees within S.

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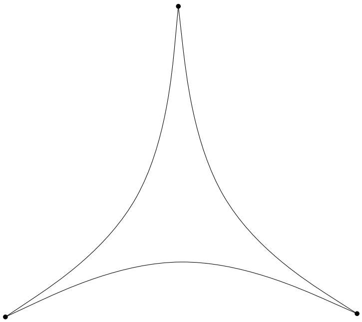
Kakeya's question (1917): Does the three-pointed deltoid shape have minimal area among such sets?



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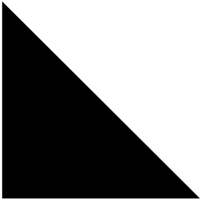

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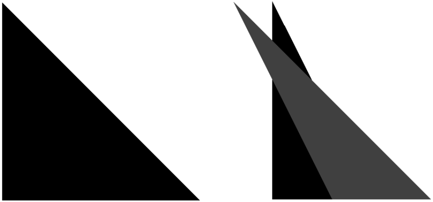

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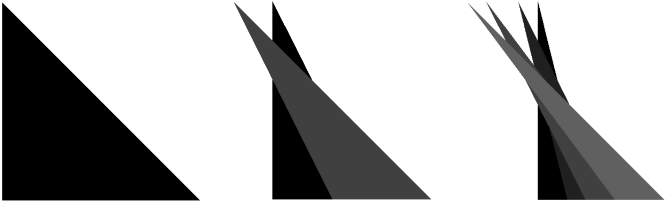

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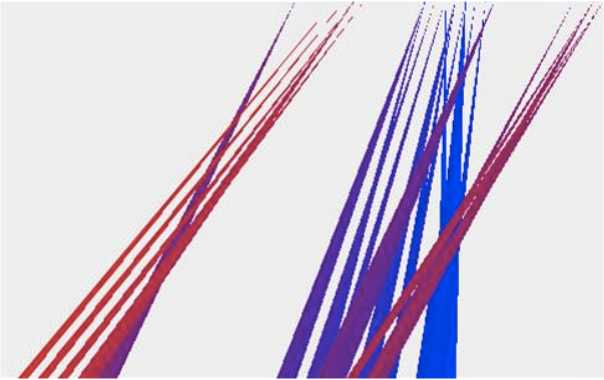

Definition of the game G_n

Two players

Two players

Two players

Hunter

Two players

Hunter

Definition of the game G_n

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Hunter

Rabbit

Two players

Where?

Hunter

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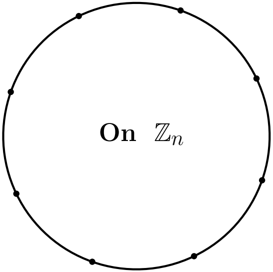

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Definition of the game

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At night – they cannot see each other....

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- We will estimate p_n, and construct a Besicovitch set of area $\asymp p_n$, that consists of $4n$ triangles.

Micah Adler, Harald Räcke, Naveen Sivadasan, Christian Sohler, and Berthold Vöcking.
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Proc. Cambridge Philos. Soc., 69:417–421, 1971.

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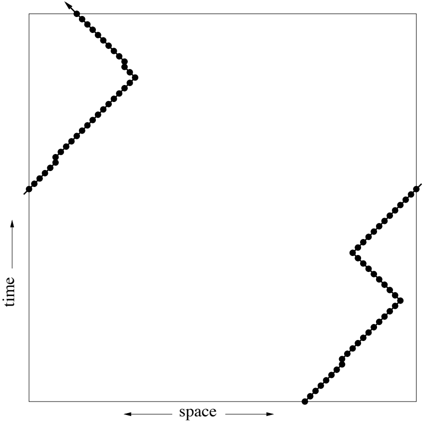
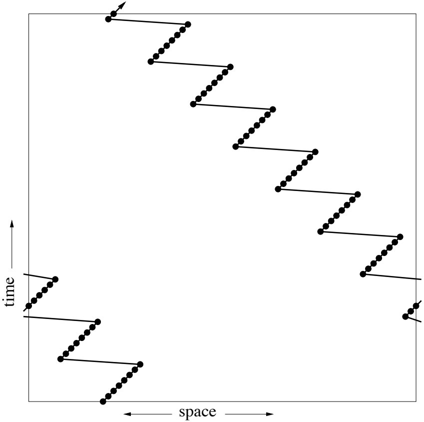
- **Zig-Zag hunter strategy:** He starts in a random direction, then switches direction with probability $1/n$ at each step.

  Rabbit counter-strategy: From a random starting node, the rabbit walks $\sqrt{n}$ steps to the right, then jumps $2\sqrt{n}$ to the left, and repeats. The probability of capture in n steps is $\asymp n^{-1/2}$.

Zig-Zag hunter strategy





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Use second moment method – calculate first and second moments of K_n.

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Extend the strategies *until time* $2n$ and *define* K_{2n} analogously.

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What about the **rabbit**? Can he **escape** for time of order $n \log n$?

Looking for a **rabbit** strategy with

$$\mathbb{P}(K_n > 0) \lesssim \frac{1}{\log n}.$$

Extend the strategies *until time* $2n$ and *define* K_{2n} analogously. Obviously

$$\mathbb{P}(K_n > 0) \leq \frac{\mathbb{E}[K_{2n}]}{\mathbb{E}[K_{2n} \mid K_n > 0]}$$

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So the **rabbit** should also choose a distribution for the jumps that favors short distances, yet grows linearly in time. This suggests a **Cauchy random walk**.

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This is what we want- **But** in the discrete setting...

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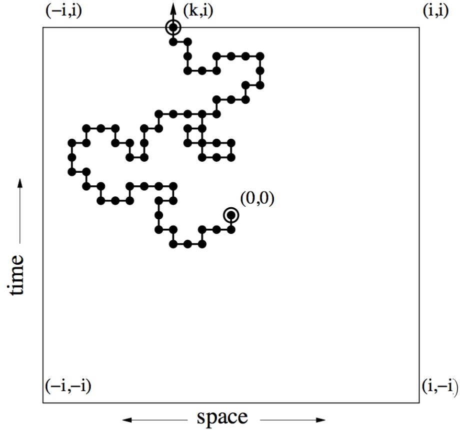

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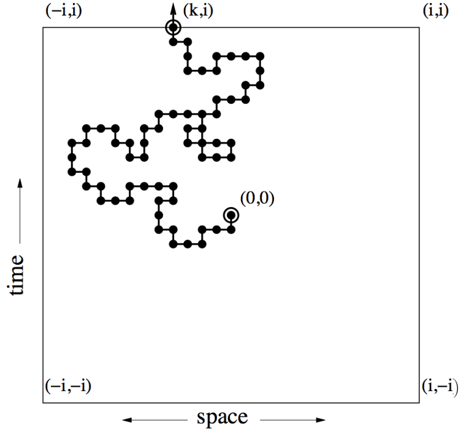

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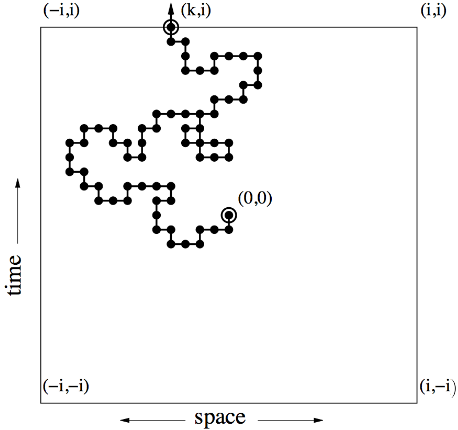

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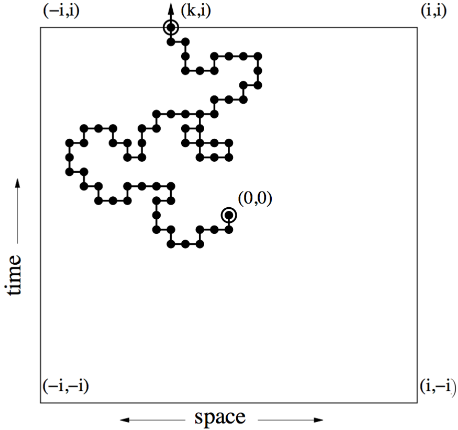
- Thus the hitting probability at $(0, i)$ is at least $1/(8i + 4)$.



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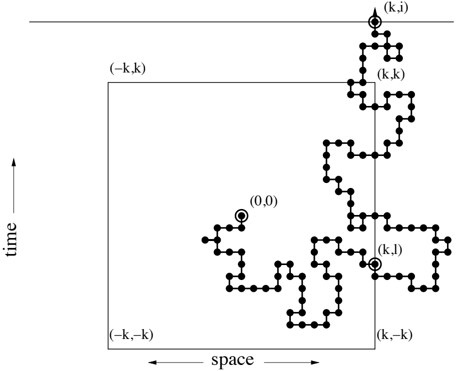

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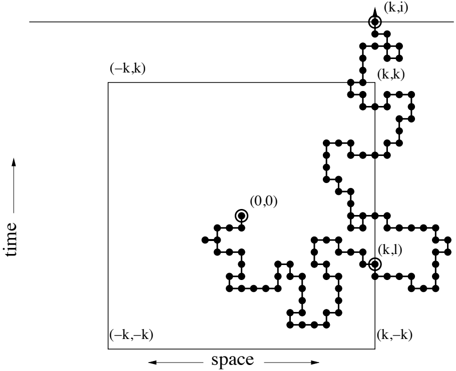
- Suppose $0 < k < i$.



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- Suppose $0 < k < i$.
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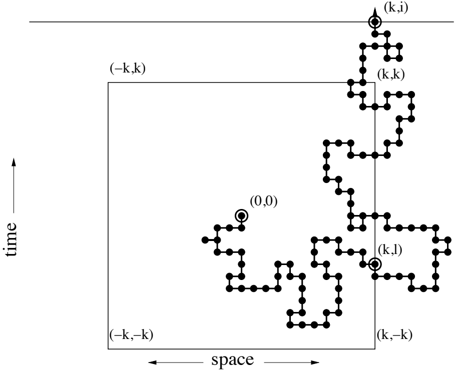

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- Suppose $0 < k < i$.
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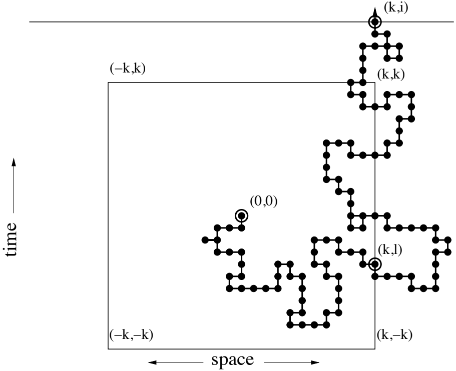
- Repeating the previous argument, the hitting probability at (k,i) is at least c/i.



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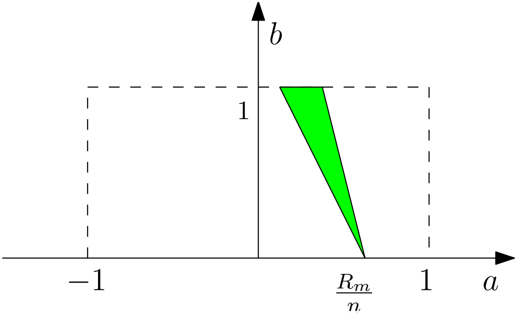

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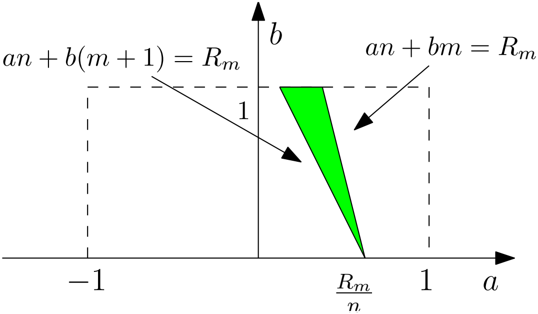
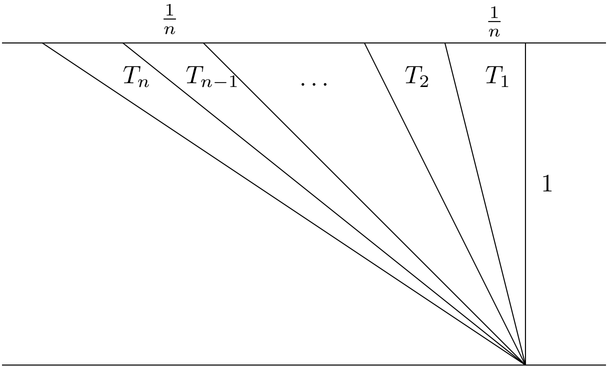


Hence the probability of collision in $[0, n)$ is half the area of the union of all such triangles, which are translates of



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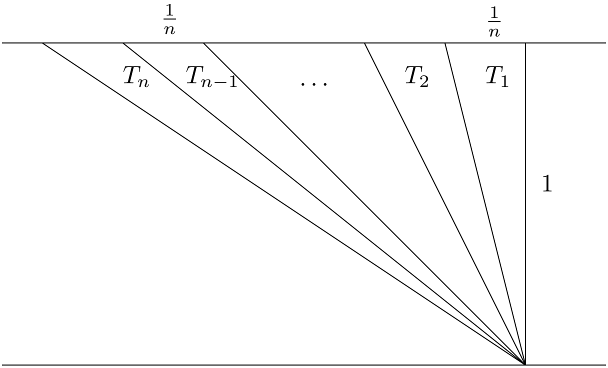

In these triangles we can find a unit segment in all directions that have an angle in $[0, \pi/4]$

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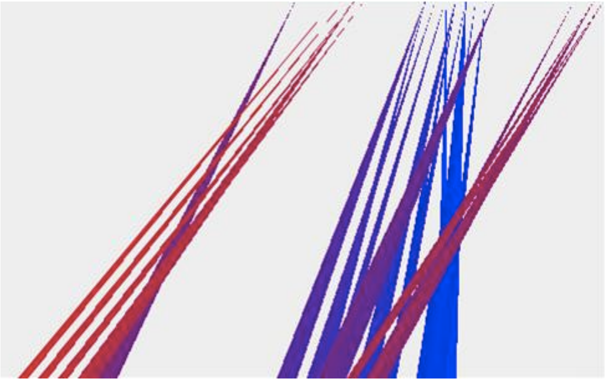

Simulation generated with $n = 32$

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$\mathrm{Leb}(\Lambda) = 0$ and **most importantly** the ε-neighbourhood satisfies almost surely

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Hence

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It is a *major open problem* whether Besicovitch sets in dimensions $d > 2$ have Hausdorff dimension equal to d.

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So the random construction is optimal.

Davies in 1971 showed that Besicovitch sets in the plane have Hausdorff dimension equal to 2.

It is a *major open problem* whether Besicovitch sets in dimensions $d > 2$ have Hausdorff dimension equal to d.

Cauchy process

The Cauchy process can be embedded in planar Brownian motion.

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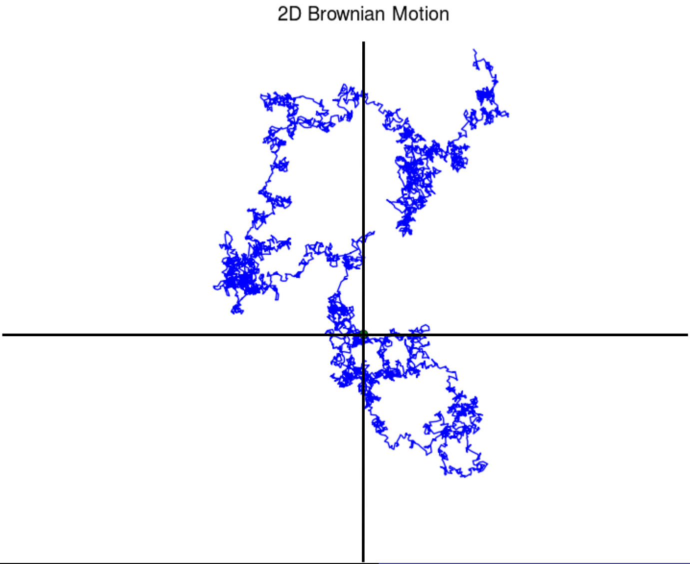

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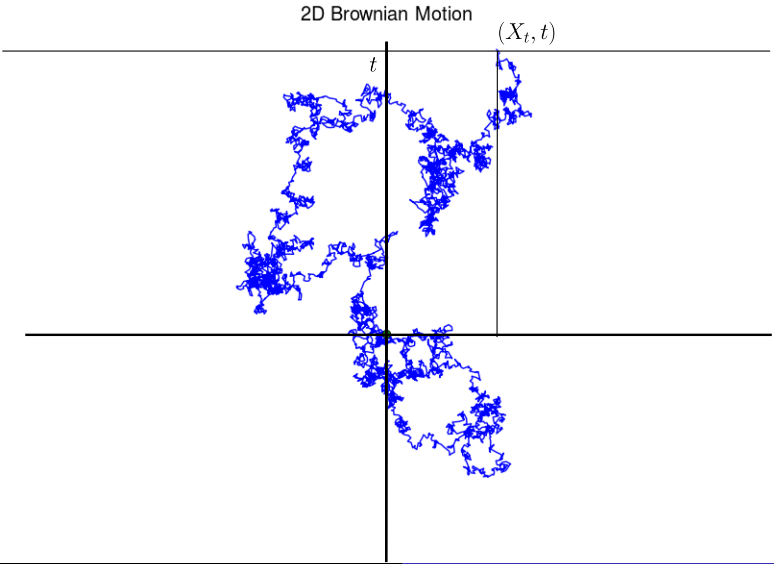

A construction from Bishop-P., *Fractals in Probability and Analysis* (cf E. Sawyer (1987))

Theorem (Besicovitch 1919, 1928) There is a set of zero area in $\mathbb{R}^2$ that contains a unit line segment in every direction.

Proof: Consider the sequence

$$\{a_k\}_{k=1}^{\infty} = \left\{0, 1, \frac{1}{2}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{7}{8}, \frac{6}{8}, \frac{5}{8}, \dots \right\},$$

i.e., $a_1 = 0$ and for $k \in [2^n, 2^{n+1})$,

$$a_k = \begin{cases} k2^{-n} - 1 & \text{if } n \text{ is even} \\ 2 - k2^{-n} & \text{if } n \text{ is odd} \end{cases}$$

Set $g(t) = t - \lfloor t \rfloor$,

$$f_k(t) = \sum_{j=2}^{k} \frac{a_{j-1} - a_j}{2^j} g(2^j t), \qquad \text{and} \qquad f(t) = \lim_{k \to \infty} f_k(t).$$

By telescoping, $f_k'(t) = -a_k$ on each component of $U = [0,1] \setminus 2^{-k}\mathbb{Z}$.

Let $K = \{(a, f(t) + at) : a, t \in [0, 1]\}$.

Fixing t shows K contains unit segments of all slopes in $[0, 1]$, so a union of four rotations of K contains unit segments of all slopes.
We need to show K has zero area.
Given $a \in [0, 1]$ and $n \geq 1$, find $k \in [2^n, 2^{n+1}]$ so that $|a - a_k| \leq 2^{-n}$. Then $f_k(t) + at$ is piecewise linear with $|\text{slopes}| \leq 2^{-n}$ on the 2^k components of $U = [0, 1] \setminus 2^{-k}\mathbb{Z}$. Hence this function maps each such component I into an interval of length at most $2^{-n}|I| = 2^{-n-k}$. Also,

$$|f(t) - f_k(t)| \leq \sum_{j=k+1}^{\infty} \frac{|a_{j-1} - a_j|}{2^j} g(2^j t) \leq 2^{-n} \sum_{j=k+1}^{\infty} 2^{-j} = 2^{-n-k} .$$

Thus $f(t) + at$ maps each component I of U into an interval of length $\leq 3 \cdot 2^{-n-k}$
Thus every vertical slice $\{t : (a, t) \in K\}$ has length zero, so by Fubini's Theorem, K has zero area.